J. Appl. Maths Mechs, Vol. 55, No. 2, pp. 266-272, 1991
Printed in Great Britain

0021-8928/91 $15.00+0.00

MODE THEORY OF ELECTROMECHANICAL ENERGY CONVERSION IN PIEZOELECTRIC MATERIALS*

O.YU. SHARII

The method of eigenfunctions is generalized to the problem of dynamic electroelasticity /1/, which allows of a larger variety of physically realisable boundary conditions. Using the method of eigenfunctions a so-called mode theory of electromagnetic energy conversion is developed. Its basic idea is the establishment of a connection between the dynamic electromechanical coupling coefficient of the specified displacement field and the mode or partial electromechanical coupling coefficients of the individual modes of oscillation. The latter, being non-linear integral characteristics of normal modes, play the same fundamental role with respect to the first as the eigenfunctions (normal modes) play with respect to the overall solution of the electro-elasticity problem. The coefficients of the expansion in eigenfunctions of the displacement field which gives a maximum permissible value of the electromechanical coupling coefficient for a given geometry, position of the electrodes, and method of supplying and removing the electrical energy.

1. For the system of equations of motion of a piezoelectric medium /2, 3/

$$c^E_{ijkl}u_{k,lj} + e_{kij}\psi_{,kj} + F_i - \rho u_i^{\cdot\cdot} = 0, \quad e_{ikl}u_{k,li} - \varepsilon^S_{ik}\psi_{,ki} = 0 \tag{1.1}$$

we will seek a solution $u_i(x, t)$, $\psi(x, t)$ in a volume V, which satisfies the initial conditions

$$u_i(x, 0) = u_{i0}(x), \quad u_i^{\cdot}(x, 0) = v_{i0}(x), \quad x \in V \tag{1.2}$$

and, at each point of the boundary $S = S_1 + S_2 = S_3 + S_4$, one of the mechanical and electrical boundary conditions

$$u_i = f_i(x, t), \ x \in S_1; \quad n_j\tau_{ij} = g_i(x, t), \ x \in S_2 \tag{1.3}$$
$$\psi = \psi_\nu(t), \quad x \in S_\nu, \quad \bigcup_{\nu=1}^{N} S_\nu = S_3; \quad n_iD_i = 0, \quad x \in S_4$$

Each of the N electrodes S_ν ($\nu = 1, 2, \ldots, N$) can be connected to a voltage generator - in this case the corresponding value of ψ_ν is known. If the value of the current through the electrode in particular, is given, the current may be zero (a passive electrode), the unknown potential $\psi_\nu(t)$ is determined from the given value (equal to zero for passive electrodes) of the total charge

$$Q_\nu \equiv -\int_{S_\nu} n_iD_i\,dS = Q_\nu(t) \tag{1.4}$$

It is best to use different systems of eigenfunctions, depending on the methods of connecting the electrodes, to solve problem (1.1)-(1.3). However, in all cases, following /1/, we have the same form of representations of the solution in the method of eigenfunctions

$$u_i = \sum_{m=1}^{\infty} U_i^{(m)}(x)q_m(t), \quad \psi = \varphi(x, t) + \sum_{m=1}^{\infty} \Psi^{(m)}(x)\,q_m(t) \tag{1.5}$$

where $U_i^{(m)}$, $\Psi^{(m)}$ are the solutions of the eigenvalue problem

$$c^E_{ijkl}U^{(m)}_{k,lj} = e_{kij}\Psi^{(m)}_{,kj} + \rho\Omega_m{}^2U_i^{(m)} = 0, \quad e_{ikl}U^{(m)}_{k,li} - \varepsilon^S_{ik}\Psi^{(m)}_{,ki} = 0 \tag{1.6}$$
$$U_i^{(m)} = 0, \ x \in S_1; \quad n_j\tau_{ij}^{(m)} = 0, \ x \in S_2$$

*Prikl. Matem. Mekhan., 55, 2, 330-337, 1991

$$\Psi_\nu^{(m)} = 0 \quad \text{or} \quad \Psi_\nu^{(m)} = \text{const}, \quad Q_\nu^{(m)} \equiv -\int_{S_\nu} n_i D_i \, dS = 0, \quad x \in S_\nu$$

$$n_i D_i^{(m)} = 0, \; x \in S_4$$

corresponding to the eigenfrequencies Ω_m, while φ is an unconnected potential, determined from the problem

$$e_{ik}^S \varphi_{,ki} = 0, \quad x \in V; \quad \varphi = \psi_\nu(t), \quad x \in S_\nu; \quad n_i e_{ik}^S \varphi_{,k} = 0, \quad x \in S_4 \tag{1.7}$$

where for the same numbers of electrodes ν, for which $Q_\nu(t)$ (1.4) and the zero $Q_\nu^{(m)}$ are given in (1.6), we have the conditions on the unbound charges $Q_\nu{}^u$

$$Q_\nu{}^u \equiv \int_{S_\nu} n_i e_{ik}^S \varphi_{,k} \, dS = Q_\nu(t) \tag{1.8}$$

Above, in addition to the notation used in /1/, we have introduced the following: $\Psi_\nu^{(m)}$ and $Q_\nu^{(m)}$ are the potential and intrinsic coupled (polarization) charge, induced by the m-th eigenfunction on the ν-th electrode, and

$$\tau_{ij}^{(m)} = c_{ijkl}^E U_{k,l}^{(m)} + e_{kij} \Psi_{,k}^{(m)}, \quad D_i^{(m)} = e_{ikl} U_{k,l}^{(m)} - e_{ik}^S \Psi_{,k}^{(m)}$$

are the voltages and induction in the volume, corresponding to the m-th eigenfunction.

The functions of time $q_m(t)$ in (1.5) are determined by the initial and boundary conditions

$$q_m(t) = q_m(0) \cos \Omega_m t + \dot{q}_m(0) \frac{\sin \Omega_m t}{\Omega_m} + \frac{1}{\Omega_m} \int_0^t \Phi_m(\tau) \sin \Omega_m (t - \tau) \, d\tau \tag{1.9}$$

$$q_m(0) = \int_V \rho u_{i0} U_i^{(m)} \, dV, \quad \dot{q}_m(0) = \int_V \rho v_{i0} U_i^{(m)} \, dV$$

$$\Phi_m(t) = \int_V F_i U_i^{(m)} \, dV - \int_{S_4} n_j \tau_{ij}^{(m)} f_i \, dS + \int_{S_4} U_i^{(m)} g_i \, dS + \sum_{\nu=1}^{N} [\psi_\nu(t) Q_\nu^{(m)} - \Psi_\nu^{(m)} Q_\nu(t)]$$

It can be seen from (1.6) and (1.9) that when choosing the corresponding system of eigenfunctions, the unknown potentials $\psi_\nu(t)$ are multiplied by the zero $Q_\nu^{(m)}$ and do not occur in the expression for $q_m(t)$.

For a later discussion it is important to note that the scheme of the method of eigenfunctions described stays the same in the case when several electrodes are short-circuited, for example, $\psi_1 = \psi_2$. The unknown quantity ψ_1 can be taken outside the summation sign in (1.9) if the system of eigenfunctions satisfies the conditions

$$\Psi_1^{(m)} = \Psi_2^{(m)}, \; Q_1^{(m)} + Q_2^{(m)} = 0 \tag{1.10}$$

All systems of eigenfunctions constructed for different methods of electrical loading, have the properties described in /1/. The choice of any of these in a specific problem is determined by the method by which the electrical energy is applied and removed.

If the piezoelectric coupling disappears, i.e. all the piezoelectric moduli e_{ikl} approach zero, then both the initial problem and the eigenfunction problem split into uncoupled mechanical and electrostatic problems. The mechanical problem, as before, has eigenvalues, and the solution for u_i retains its form, similar to the representation from the pure theory of elasticity. Nevertheless, the electrostatic "eigenvalue problem" does not contain a spectral parameter and does not have eigenfunctions, as a result of which all the $\Psi^{(m)}$ vanish. Hence, in its physical meaning and in accordance with the notation, as well as the method used in the method of eigenfunctions, the function $\varphi(x, t)$ is a complete solution of the uncoupled electrostatic problem.

This formulation of the method of eigenfunctions also holds in the case when some of the electrodes are situated in the volume of the piezoelectric material - it is only necessary to replace the formulae for the charges (1.4), (1.6) and (1.8), by calculating them from the change in electric induction on passing through the electrodes /2, 4/.

Substituting the expansions (1.5) into the formula for the potential energy of a piezoelectric medium /2, 3/

$$U = \frac{1}{2}\int_V (\tau_{ij}e_{ij} + E_k D_k)\,dV$$

using the orthogonality of the eigenfunctions of the displacements /1/ and denoting by φ_ν the value of the free potential φ on the ν-th electrode, we obtain after the necessary reduction

$$U = \frac{1}{2}\sum_{m=1}^{\infty}\Omega_m^2 q_m^2 + \sum_{\nu=1}^{N}\varphi_\nu Q_\nu^u - \frac{1}{2}\sum_{\nu=1}^{N}\varphi_\nu Q_\nu^u \tag{1.11}$$

The expression for the kinetic energy has the form

$$T \equiv \frac{1}{2}\int_V \rho u_i^* u_i^*\,dV = \frac{1}{2}\sum_{m=1}^{\infty} q_m^{*2}$$

2. To calculate the dynamic electromechanical coupling coefficient using the eigenfunction method we need to have relations between the values of the unbound charges Q_ν^u and the values of φ_ν. Since the field of the free potential is electrostatic, these relations have the form

$$Q_\mu^u = \sum_{\nu=1}^{N} c_{\mu\nu}\varphi_\nu, \quad \mu = 1, 2, \ldots, N$$

where $\|c_{\mu\nu}\|$ is a symmetrical $(N \times N)$-matrix of the static capacitances /5/, the components of which can be found from the problem of the electrostatic loading of a piezoelectric material for constant strains, and can be assumed to be known.

The sum of the unbound charges on the electrodes is always zero

$$\sum_{\mu=1}^{N} Q_\mu^u = 0 \tag{2.1}$$

This follows from the Laplace Eq.(1.7), integrated over the volume V, and occurs as part of our assumption that there are no electric scattering fields outside the material /2, 4/. Since condition (2.1) holds for any φ_ν, it follows that the capacitance matrix

$$\sum_{\nu=1}^{N} c_{\mu\nu} = 0, \quad \mu = 1, 2, \ldots, N$$

is degenerate.

For example, for a two-electrode system $(N = 2)$ the symmetrical degenerate matrix contains one independent element, called the capacitance C

$$\|c_{\mu\nu}\| = \begin{Vmatrix} C & -C \\ -C & C \end{Vmatrix}$$

while the unbound charges are proportional to the potential differences

$$Q_1^u = -Q_2^u = C(\varphi_1 - \varphi_2) \tag{2.2}$$

3. Consider a two-electrode piezoelectric material, the distribution of the strains in which is completely defined by the expansion coefficients q_m of $U_i^{(m)}$ in a system of eigenfunctions, obtained for short-circuited electrodes, i.e., when (1.10) is satisfied for all m. Following the energy definition /2/, for the electroelastic field with displacements

$$u_i = \sum U_i^{(m)} q_m \tag{3.1}$$

we obtain the potential energy for open-circuited electrodes (U^{oc}) and short-circuited electrodes (U^{sc}) and then the dynamic electromechanical coupling coefficient k_d^2 from the formula

$$k_d^2 = (U^{oc} - U^{sc})/U^{oc} \tag{3.2}$$

Here and everywhere later, unless otherwise stated, the summation is over m from $m = 1$ to ∞.

According to the eigenfunction method, the solution for the electric potential ψ is given by the second equation of (1.5) and as $\Psi^{(m)}$ and q_m are known, the values of φ for the open-circuit and short-circuit states are completely defined. However, to calculate k_d^2 we only need to know the difference $\varphi_1 - \varphi_2$. We will write (1.11), taking (2.2) into account in the form

$$U = \frac{1}{2}\sum \Omega_m^2 q_m^2 + (\psi_1 - \psi_2)\, Q_1^u - \frac{1}{2}(\varphi_1 - \varphi_2)\, Q_1^u \tag{3.3}$$

For open-circuited electrodes $(Q_1 = Q_2 = 0)$, it follows from the second equation of (1.5) and from (1.10) that

$$\psi_1 - \psi_2 = \varphi_1 - \varphi_2 \tag{3.4}$$

while the relation obtained from (1.5)

$$Q_\nu = Q_\nu^u + \sum Q_\nu^{(m)} q_m \tag{3.5}$$

gives

$$Q_1^u = -\sum Q_1^{(m)} q_m \tag{3.6}$$

Substituting (3.4) into (3.3) and using (2.2) and (3.6), we obtain

$$U^{oc} = \frac{1}{2}\sum \Omega_m^2 q_m^2 + \frac{1}{2} C^{-1} \Big(\sum Q_1^{(m)} q_m\Big)^2 \tag{3.7}$$

Assuming now that the electrodes are short-circuited $(\psi_1 - \psi_2 = 0)$, we obtain from (1.5) and (1.10) $\varphi_1 - \varphi_2 = 0$, and (3.3) leads to the equation

$$U^{sc} = \frac{1}{2}\sum \Omega_m^2 q_m^2 \tag{3.8}$$

From (3.2), (3.7) and (3.8) we obtain

$$k_d^2/(1 - k_d^2) = \Big(\sum Q_1^{(m)} q_m\Big)^2 \Big(C \sum \Omega_m^2 q_m^2\Big)^{-1} \tag{3.9}$$

The values of the electromechanical coupling coefficient for the m-th mode of oscillation k_m^2 can be obtained by assuming that all q_m in expansion (3.1), apart from one, are zero. These partial or mode electromechanical coupling coefficients are given by the equations

$$k_m^2/(1 - k_m^2) = Q_1^{(m)2} (C\Omega_m^2)^{-1}, \quad m = 1, 2, \ldots \tag{3.10}$$

Eliminating from the last two equations the values of the natural charges $Q_1^{(m)}$, we obtain the required relationship between the electromechanical coupling coefficient for arbitrary strain and the electromechanical coupling coefficients of the individual normal modes

$$k_d^2/(1 - k_d^2) = \Big(\sum \Omega_m q_m k_m / \sqrt{1 - k_m^2}\Big)^2 \Big(\sum \Omega_m^2 q_m^2\Big)^{-1} \tag{3.11}$$

The last equation enables us to estimate the maximum electromechanical coupling coefficient obtained for a given arrangement of the electrodes: from Cauchy's inequality

$$k_d^2/(1 - k_d^2) \leqslant \sum k_m^2/(1 - k_m^2) \tag{3.12}$$

The equality sign in (3.12) holds when $\Omega_m q_m = A k_m / \sqrt{1 - k_m^2}$, where A is an arbitrary constant, independent of m. The corresponding maximum value of the electromechanical coupling coefficient (the left-hand side of (3.12) is an increasing function of k_d^2) is reached for displacements

$$u_i = A \sum \Omega_m^{-1} (k_m / \sqrt{1 - k_m^2}) U_i^{(m)}(x) \tag{3.13}$$

As follows from Sect.1, the displacement field can be represented in the form of an expansion in another system of eigenfunctions, obtained for open-circuited electrodes, when (1.10) is replaced by the equations

$$Q_1^{(m)} = 0, \quad Q_2^{(m)} = 0 \tag{3.14}$$

Suppose (3.1) represents this expansion; then for open-circuited electrodes ($Q_1 = Q_2 = 0_{oc}$) it follows from (3.14) and (3.5) that $Q_1^{\prime\prime} = 0$, and (3.3) gives for U^{oc} an expression identical in form with (3.8), but with other values of Ω_m and q_m.

For short-circuited electrodes, by setting up the difference $\varphi_1 - \varphi_2$, by using the second equation of (1.5), when $\psi_1 - \psi_2 = 0$ we obtain the relation

$$\varphi_1 - \varphi_2 = -\sum (\Psi_1^{(m)} - \Psi_2^{(m)}) q_m$$

the substitution of which, in combination with (2.2), into (3.3) gives U^{sc}, while the electromechanical coupling coefficient calculated from (3.2) is

$$k_d^2 = C \left[\sum (\Psi_1^{(m)} - \Psi_2^{(m)}) q_m\right]^2 \left(\sum \Omega_m^2 q_m^2\right)^{-1}$$

Now setting up, in the same way as (3.10), an expression for the partial electromechanical coupling coefficient k_m^2 and eliminating $\Psi_1^{(m)} - \Psi_2^{(m)}$, we obtain the relation between k_d^2 and the electromechanical coupling coefficient of the normal modes, corresponding to open-circuited electrodes

$$k_d^2 = \left(\sum \Omega_m q_m k_m\right)^2 \left(\sum \Omega_m^2 q_m^2\right)^{-1} \tag{3.15}$$

By estimating the right-side of (3.15) using the Cauchy inequality, we obtain in this case

$$k_d^2 \leqslant \sum k_m^2 \tag{3.16}$$

while the displacements which give the maximum electromechanical coupling coefficient (the equality sign in (3.16)) can be represented by the expansion in eigenfunctions

$$u_i = A \sum \Omega_m^{-1} k_m U_i^{(m)}(x)$$

The basic idea of the energy method of determining the dynamic electromechanical coupling coefficients is the fact that this characteristic should be completely defined by the deformation (displacement) field in the volume of the body and the arrangement of the electrodes /2/. Definition (3.2) is comprehensive in the case of two electrodes due to the fact that there is only one method of converting a piezoelectric body from an electrically open-circuit state to a short-circuit state. The use of this for a multi-electrode body gives an ambiguous result. Thus, for a three-electrode body we can arrive at a state with three short-circuited electrodes by connecting a third electrode to the two previously short-circuited electrodes, or simultaneously short-circuiting all three electrodes. The values of k_d^2 calculated from (3.2) turn out in this case to be different.

There is no definition of the electromechanical coupling coefficients of a multi-electrode body, going by the literature /2, 3/, and to introduce it we were guided by the following considerations. For the appropriately calculated electromechanical coupling coefficients it is desirable that the structure of (3.11) and (3.15), containing only eigenfrequencies, the coefficients of the expansions in eigenfunctions and the partial electromechanical coupling coefficients should be preserved, i.e. it should not depend on the number of electrodes. Here it should be noted that they can be obtained by simultaneous elimination, for example, from (3.9) and (3.10), of both the bound charges $Q_1^{(m)}$, and the capacitance C. For an N-electrode body $(N \geqslant 3)$ to change from a completely open-circuited state to a completely short-circuited state in the expression for k_d^2 and k_m^2 we take into account $^1/_2 N (N-1)$ independent components of the capacitance matrix and $N-1$ charges $Q_v^{(m)}$ which makes such simultaneous elimination impossible.

On the other hand, in the majority of piezoelectronic devices /3/, the energy at each instant of time is removed from a single pair of electrodes, where the idea of an "electrode" includes several constructively short-circuited electrodes. Thus, the electromechanical coupling coefficients of a $(2n+2)$-electrode cylinder /2/ is in fact calculated on the assumption that no redistribution of the potentials occurs when the electrodes are successively short-circuited, i.e. they are short-circuited through one, as occurs in the multilayer transducers used in practice. Hence, such a transducer is essentially a two-electrode device. Moreover, one must take into account in the definition of the electromechanical coupling coefficient that the energy to be converted is extracted by short-circuiting a pair of electrodes, and the possible redistribution of energy between the body and the generators through the remaining electrodes must be eliminated from consideration.

By turning our attention to the physically realizable methods of extracting electrical energy, the electromechanical coupling coefficients of a multi-electrode body will be calculated from (3.2) for a pair of electrodes, assumed to be open-circuited or short-circuited for specified displacements in the volume. Here we require to preserve the zero values of the

potential on the electrodes the voltage connected to the generators and the zero on the electrodes supplied by the current generators or on the passive electrodes. Writing the theorem on the change in total energy (/4/, Eq.(25.6)), for zero velocities, i.e. unchanged strains in the volume

$$\frac{dU}{dt} = \sum_{\nu=1}^{N} \psi_\nu Q_\nu^{\cdot}$$

we see that the flow of energy through the electrodes with zero ψ_ν or Q_ν is zero.

Consider an N-electrode body in which the electrodes with numbers $\nu = 3, 4, \ldots, M$ are supplied by voltage generators, while those with numbers $\nu = M+1, M+2, \ldots, N$ are supplied by current generators or are passive. We will obtain a solution of the initial-boundary value problem (1.1)-(1.4) in the form (1.5). Here we will use the system of eigenfunctions, defined from the conditions

$$\psi_\nu^{(m)} = 0,\ \nu = 3, 4, \ldots, M;\ Q_\nu^{(m)} = 0,\ \nu = M+1, M+2, \ldots, N$$

On the first two electrodes, for which the electromechanical coupling coefficient is calculated, the eigenfunctions satisfy either (1.10) or (3.14).

Keeping, at the instant when the body is transferred from the open-circuit to the short-circuit state,

$$\psi_\nu = 0,\ \nu = 3, 4, \ldots, M;\ Q_\nu = 0,\ \nu = M+1, M+2, \ldots, N$$

we obtain

$$\varphi_\nu = 0,\ \nu = 3, 4, \ldots, M;\ Q_\nu^{u} = 0,\ \nu = M+1, M+2, \ldots, N$$

and the formula for the energy (1.11) takes the form

$$U = \frac{1}{2}\sum_{(m)} \Omega_m^2 q_m^2 + (\psi_1 Q_1^{u} + \psi_2 Q_2^{u}) - \frac{1}{2}(\varphi_1 Q_1^{u} + \varphi_2 Q_2^{u})$$

Assuming $Q_1 = Q_2 = 0$, we obtain U^{oc}. When calculating U^{sc}, in addition to $\psi_1 - \psi_2 = 0$ we must use the condition $Q_1 + Q_2 = 0$, which denotes that no extraneous charges have been introduced. Note that for $N = 2$ the last equation is satisfied automatically, and hence is not formulated in explicit form.

Just as for a two-electrode body, the quantities Q_1^{u} and $\varphi_1 - \varphi_2$ turn out to be connected by relations (2.2) with the sole difference that C now denotes a certain positive combination of elements of the capacitance matrix. Hence, the formulas for k_d^2 and k_m^2 and the relations between them retain their form.

Of the two possible methods of electrical excitation of the first and second electrodes, one is usually employed in practice. Corresponding to this, in specific problems the maximum achievable electromechanical coupling coefficient can be found either from (3.12) or from (3.16), but the result of course is the same. The choice of the conditions for the eigenfunctions on the first and second electrodes is made from considerations of simplicity in finding the eigenfunctions (see the example).

Formulas (3.12) or (3.16) also enable us to estimate the maximum electromechanical coupling coefficients when the number of excited modes is limited, and to determine the ratio of the amplitudes of the individual modes which give the value of the electromechanical coupling coefficient that is closest to the maximum, including in non-stationary problems also.

4. The application of the proposed theory to determine the maximum electromechanical coupling coefficients and the corresponding distributions of displacements to a longitudinally and transversely polarized rod, and also to a piezoceramic disc that is thickness-polarized confirmed well-known results: these electromechanical coupling coefficients are equal to the static values k_{33}^2, k_{31}^2 and k_p^2 /2/ and they are obtained in uniform deformations.

A more complex example, in which the result was not known in advance, requires a consideration of a four-electrode disc piezoelectric transducer of thickness h and radius a with a free contour, the continuous electrodes of which are cut over circles of radius $b < a$. Suppose the piezoelectric transducer is excited by a potential difference applied to the central electrodes $0 < r < b$, and the energy is removed from ring electrodes $b < r < a$. We will determine the partial (k_m^2) and maximum electromechanical coupling coefficients for constantly short-circuited central electrodes.

It is obvious that in this problem a simpler system of eigenfunctions will be that obtained for short-circuited external electrodes: they are identical with the eigenfunctions of a disc with continuous electrodes. The eigenfrequencies $\Omega_m = \varkappa_m c$, where c is the plate

velocity (/2/, p.99), are defined by the positive roots of the equation

$$\varkappa a J_0(\varkappa a) - (1-\nu) J_1(\varkappa a) = 0 \tag{4.1}$$

while the orthonormalized eigenfunctions /1/ are given by

$$U_m = (2\pi\rho h)^{-1/2} \varkappa_m \lambda_m J_1(\varkappa_m r)$$
$$\lambda_m = [1/2 (\varkappa_m a)^2 (J_0^2(\varkappa_m a) + J_1^2(\varkappa_m a)) - (1-\nu) J_1^2(\varkappa_m a)]^{-1/2}$$

The maximum achievable electromechanical coupling coefficient is given by the sum of the series

$$\left(\frac{k^2}{1-k^2}\right)_{\max} = \frac{1+\nu}{a^2-b^2} \frac{k_p^2}{1-k_p^2} \sum \lambda_m [aJ_1(\varkappa_m a) - bJ_1(\varkappa_m b)]^2 \tag{4.2}$$

the terms of which are equal to $k_m^2/(1-k_m^2)$. This is achieved for displacements u calculated according to the rule (3.13) (an unimportant constant factor is omitted)

$$u = \Sigma \lambda_m [aJ_1(\varkappa_m a) - bJ_1(\varkappa_m b)] J_1(\varkappa_m r) \tag{4.3}$$

Rewriting the frequency Eq.(4.1) in the form $\varkappa a J_1'(\varkappa a) + \nu J_1(\varkappa a) = 0$, it can be shown that (4.3) is the expansion of the required displacement in a Diny series /6/, the sum of which is equal to

$$u = \frac{1}{2}\left[\frac{1-\nu}{1+\nu}(1-\varepsilon^2) r + \left(r - \varepsilon^2 \frac{a^2}{r}\right) H(r-b)\right], \quad \varepsilon = \frac{b}{a}$$

where H is the Heaviside unit function.

The displacement field which provides a maximum electromechanical coupling coefficient under the internal electrodes $(0 < r < b)$ is a linear function, while that under the ring electrodes $(b < r < a)$ is a combination of a linear function and a function inversely proportional to r. It would have been difficult to foresee this from elementary considerations.

Comparing (4.2) and (4.3) we see that the first sum is proportional to $au(a) - bu(b)$, whence we obtain

$$k_{\max}^2 = k_p^2 \left(1 - \frac{1-\nu}{2}\varepsilon^2\right)\left(1 - \frac{1-\nu}{2} k_p^2 \varepsilon^2\right)^{-1}$$

The right-hand side of the last equation is a decreasing function of the geometrical parameter ε. The limiting value $k_{\max}^2$ for the material *PZT-4* /2/, $\nu = 0.3$, is $k_p^2 = 0.34$ for $\varepsilon = 0$ (uniform deformation of a disc with two continuous electrodes) is equal to k_p^2 for $\varepsilon = 1$ (infinitely narrow ring electrodes), and $k_{\max}^2$ is a minimum and equal to $0.74\, k_p^2$.

The author thanks A.F. Ulitko for discussing the results.

REFERENCES

1. ZHARII O.YU., The method of expansion in eigenfunctions in problems of dynamic electroelasticity, PMM, 54, 1, 1990.
2. GRINCHENKO V.T., ULITKO A.F. and SHUL'GA N.A., Electroelasticity, Nauk. Dumka, Kiev, 1989.
3. PARTON V.Z. and KUDRYAVTSEV B.A., Electromagnetic Elasticity of Piezoelectric and Electrically Conducting Bodies, Nauka, Moscow, 1988.
4. ZHARII O.YU. and ULITKO A.F., Introduction to the Mechanics of Non-stationary Vibrations and Waves, Vishcha Shkola, Kiev, 1989.
5. SMYTHE W., Electrostatics and Electrodynamics, IIL Moscow, 1954.
6. BATEMAN H. and ERDELYI A., Higher Transcendental Functions, 2, Nauka, Moscow, 1974.

Translated by R.C.G.